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Implications of $Tr\gamma_5 = 0$ in Lattice Gauge Theory[1]

Kazuo Fujikawa

Department of Physics, University of Tokyo
Bunkyo-ku, Tokyo 113, Japan

Abstract

We analyze the implications of the relation $Tr\gamma_5 = 0$, which is customarily assumed in practical lattice calculations. On the basis of the finite dimensional representations of the Ginsparg-Wilson algebra, it is shown that this relation reflects the species doubling in lattice theory; topological excitations associated with species doublers, which have eigenvalue $2/a$, contribute to $Tr\gamma_5$ without any suppression. In this sense, the relation $Tr\gamma_5 = 0$ is valid only when we allow the presence of unphysical states in the Hilbert space; this statement is also valid in the Pauli-Villars regularization. If one eliminates the contributions of the unpysical states, the trace $Tr\gamma_5$ is replaced by $Tr\Gamma_5 \equiv Tr\gamma_5(1-\frac{1}{2}aD)$ which gives rise to the Pontryagin index, to be consistent with the continuum analysis.

1 Introduction

Recent developments in the treatment of fermions in lattice gauge theory led to a better understanding of chiral symmetry[1]-[9]. These developments are based on a lattice Dirac operator D which satisfies the so-called Ginsparg-Wilson relation[1]

$$\gamma_5 D + D\gamma_5 = aD\gamma_5 D \tag{1.1}$$

where γ_5 is a hermitian chiral Dirac matrix. An explicit example of the operator satisfying (1.1) and free of species doubling has been given by Neuberger[2]. The operator has also been discussed as a fixed point form of block transformations [3]. The relation (1.1) led to the interesting analyses of the notion of index in lattice gauge theory[4]-[9].

The index relation is often written as [4][5]

$$-Tr\gamma_5 \frac{1}{2}aD = n_+ - n_- \tag{1.2}$$

where $n_\pm$ stand for the number of normalizable zero modes in

$$D\varphi_n = 0 \tag{1.3}$$

with simultaneous eigenvalues $\gamma_5\varphi_n = \pm\varphi_n$.

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In the continuum path integral treatment of chiral anomaly, the relation

$$Tr\gamma_5 = n_+ - n_- \tag{1.4}$$

in a suitably regularized sense (with help of a local version of the index theorem[10]) plays a fundamental role[11]. On the other hand, it is expected that the relation

$$Tr\gamma_5 = 0 \tag{1.5}$$

holds on a finite lattice. As Chiu correctly pointed out[12], this relation (1.5) leads to an interesting *constraint*

$$Tr\gamma_5 = n_+ - n_- + N_+ - N_- = 0 \tag{1.6}$$

where $N_\pm$ stand for the number of eigenstates $D\varphi_n = (2/a)\varphi_n$ with $\gamma_5\varphi_n = \pm\varphi_n$, respectively.

On the basis of the work [13], we here argue that $Tr\gamma_5 = 0$ implies the inevitable contribution from unphysical (would-be) species doublers in lattice theory or an unphysical bosonic spinor in Pauli-Villars regularization. In other words, $Tr\gamma_5 = 0$ cannot hold in the physical Hilbert space consisting of physical states only, and the continuum limit of $Tr\gamma_5 = 0$ is not defined consistently, as is seen in (1.6). It is also shown that the failure of the decoupling of heavy fermions in the anomaly calculation is crucial to understand the consistency of the customary lattice calculation of anomaly where $Tr\gamma_5 = 0$ is used. (The continuum limit in this paper stands for the so-called "naive" continuum limit with $a \to 0$, and the lattice size is gradually extended to infinity for any finite a in the process of taking the limit $a \to 0$.)

2 Finite dimensional representations of the Ginsparg - Wilson algebra

We start with the finite dimensional representations[13] of the basic algebraic relation (1.1). A construction of the operator D, which satisfies the Ginsparg-Wilson relation on a finite lattice, by using a corresponding operator D on an infinite lattice has been discussed in Ref.[14]. We first define an operator

$$\Gamma_5 \equiv \gamma_5(1 - \frac{1}{2}aD) \tag{2.1}$$

which is hermitian and satisfies the basic relation derived from (1.1)

$$\Gamma_5\gamma_5 D + \gamma_5 D\Gamma_5 = 0. \tag{2.2}$$

Namely, Γ_5 plays a role of γ_5 in continuum theory. This relation suggests that if

$$\gamma_5 D\phi_n = \lambda_n\phi_n, \quad (\phi_n, \phi_n) = 1 \tag{2.3}$$

for the *hermitian* operator $\gamma_5 D$, then

$$\gamma_5 D(\Gamma_5\phi_n) = -\lambda_n(\Gamma_5\phi_n). \tag{2.4}$$

Namely, the eigenvalues λ_n and $-\lambda_n$ are always paired if $\lambda_n \neq 0$ and $(\Gamma_5\phi_n, \Gamma_5\phi_n) \neq 0$. The inner product $\phi_n^\dagger\phi_n = (\phi_n, \phi_n) \equiv \sum_x a^4 \phi_n^\star(x)\phi_n(x)$ is defined by summing over all the lattice points, which are not explicitly written in ϕ_n.

We evaluate the norm of $\Gamma_5\phi_n$

$$\begin{aligned}
(\Gamma_5\phi_n, \Gamma_5\phi_n) &= (\phi_n, (\gamma_5 - \frac{a}{2}\gamma_5 D)(\gamma_5 - \frac{a}{2}\gamma_5 D)\phi_n) \\
&= (\phi_n, (1 - \frac{a}{2}\gamma_5(\gamma_5 D + D\gamma_5) + \frac{a^2}{4}(\gamma_5 D)^2)\phi_n) \\
&= (\phi_n, (1 - \frac{a^2}{4}(\gamma_5 D)^2)\phi_n) \\
&= (1 - \frac{a}{2}\lambda_n)(1 + \frac{a}{2}\lambda_n). \qquad (2.5)
\end{aligned}$$

Namely ϕ_n is a "highest" state

$$\Gamma_5\phi_n = (\gamma_5 - \frac{a}{2}\gamma_5 D)\phi_n = 0 \qquad (2.6)$$

if $(1 - \frac{a}{2}\lambda_n)(1 + \frac{a}{2}\lambda_n) = 0$ for the Euclidean $SO(4)$- invariant positive definite inner product (ϕ_n, ϕ_n). We thus conclude that the states ϕ_n with $\lambda_n = \pm\frac{2}{a}$ are *not* paired by the operation $\Gamma_5\phi_n$ and are the simultaneous eigenstates of γ_5, $\gamma_5\phi_n = \pm\phi_n$ respectively. These eigenvalues λ_n are also the maximum or minimum of the possible eigenvalues of $\gamma_5 D$. This is based on the relation

$$\phi_n^\dagger\gamma_5\phi_n = \frac{a}{2}\lambda_n\phi_n^\dagger\phi_n = \frac{a}{2}\lambda_n \qquad (2.7)$$

for $\lambda_n \neq 0$, which is derived by sandwiching the relation(1.1) by $\phi_n^\dagger\gamma_5$ and ϕ_n.Namely, $|\frac{a\lambda_n}{2}| = |\phi_n^\dagger\gamma_5\phi_n| \leq ||\phi_n||||\gamma_5\phi_n|| = 1$.

For the vanishing eigenvalues $\gamma_5 D\phi_n = 0$, one can show that $\gamma_5 D\gamma_5\phi_n = 0$ by using the relation (1.1). Thus one can choose the chiral eigenstates $\gamma_5 D[(1 \pm \gamma_5)/2]\phi_n = 0$. Namely

$$\gamma_5 D\phi_n = 0, \quad \gamma_5\phi_n = \pm\phi_n. \qquad (2.8)$$

All the remaining normalizable states with $0 < |\lambda_n| < 2/a$, which appear pairwise with $\lambda_n = \pm|\lambda_n|$ (note that $\Gamma_5(\Gamma_5\phi_n) = (1 - (a\lambda_n/2)^2)\phi_n \propto \phi_n$ for $|a\lambda_n/2| \neq 1$) , satisfy the relations

$$\begin{aligned}
\phi_n^\dagger\Gamma_5\phi_n &= 0, \\
\phi_n^\dagger\gamma_5\phi_n &= \frac{a\lambda_n}{2}, \\
\phi_m^\dagger\gamma_5\phi_n &= 0 \;\; for \;\; \lambda_m \neq \lambda_n, \;\; \lambda_m\lambda_n > 0. \qquad (2.9)
\end{aligned}$$

These states ϕ_n cannot be the eigenstates of γ_5 as $|a\lambda_n/2| < 1$.

The index (1.2) in the present context is most naturally defined as

$$\begin{aligned}
Tr\Gamma_5 &= \sum_n \phi_n^\dagger\Gamma_5\phi_n \\
&= \sum_{\lambda_n=0} \phi_n^\dagger\Gamma_5\phi_n + \sum_{0<|\lambda_n|<2/a} \phi_n^\dagger\Gamma_5\phi_n + \sum_{\lambda_n=\pm 2/a} \phi_n^\dagger\Gamma_5\phi_n \\
&= \sum_{\lambda_n=0} \phi_n^\dagger\gamma_5\phi_n = n_+ - n_- \qquad (2.10)
\end{aligned}$$

On the other hand, the index relation commonly written in the form [4]-[6]

$$\begin{aligned} Tr(\frac{-1}{2}a\gamma_5 D) &= -\sum_n \phi_n^\dagger \frac{a\gamma_5}{2} D\phi_n \\ &= -\frac{a}{2}\sum_n \lambda_n = -N_+ + N_- \end{aligned} \tag{2.11}$$

is saturated by the states $N_\pm$,where $N_\pm$ stand for the number of isolated (un-paired) states with $\lambda_n = \pm 2/a$ and $\gamma_5\phi_n = \pm\phi_n$, respectively.

The relation $Tr\gamma_5 = 0$, which is expected to be valid on a finite lattice, leads to (by using (2.7))

$$\begin{aligned} Tr\gamma_5 &= \sum_n \phi_n^\dagger \gamma_5 \phi_n \\ &= \sum_{\lambda_n=0} \phi_n^\dagger \gamma_5 \phi_n + \sum_{\lambda_n\neq 0} \phi_n^\dagger \gamma_5 \phi_n \\ &= \sum_{\lambda_n=0} \phi_n^\dagger \gamma_5 \phi_n + \sum_{\lambda_n\neq 0} \frac{a}{2}\lambda_n \\ &= n_+ - n_- + \sum_{\lambda_n\neq 0} \frac{a}{2}\lambda_n = 0. \end{aligned} \tag{2.12}$$

In the last line of this relation, all the states except for the states with $\lambda_n = \pm 2/a$ cancel pairwise for $\lambda_n \neq 0$. We thus obtain a chirality sum rule $n_+ - n_- + N_+ - N_- = 0$ [12] or,

$$n_+ + N_+ = n_- + N_-. \tag{2.13}$$

These relations show that the chirality asymmetry at vanishing eigenvalues is balanced by the chirality asymmetry at the largest eigenvalues with $|\lambda_n| = 2/a$.

Those properties we analyzed so far in this Section hold both for non-Abelian and Abelian gauge theories. We did not specify precise boundary conditions, since our analysis is valid once non-trivial zero modes appear for a given boundary condition. For an Abelian theory, one needs to introduce the gauge field configuration with suitable boundary conditions, which carries a non-vanishing magnetic flux , to generate a non-trivial index $n_+ - n_-$ [14]. Our analysis of the index in this Appendix is formal, since it is well known that the Ginsparg-Wilson relation (1.1) by itself does not uniquely specify the index or the coefficient of chiral anomaly for a given gauge field configuration [15].A suitable choice of parameters in the overlap operator [2], for example, uniquely specifies the index.

To summarize the analyses of the present Section, all the normalizable eigenstates ϕ_n of the hermitian $\gamma_5 D$ on a finite lattice are categorized into the following 3 classes:
(i) $n_\pm$ states,

$$\gamma_5 D\phi_n = 0, \quad \gamma_5\phi_n = \pm\phi_n, \tag{2.14}$$

(ii) $N_\pm$ states($\Gamma_5\phi_n = 0$),

$$\gamma_5 D\phi_n = \pm\frac{2}{a}\phi_n, \quad \gamma_5\phi_n = \pm\phi_n, \quad \textit{respectively}, \tag{2.15}$$

(iii) Remaining states with $0 < |\lambda_n| < 2/a$,

$$\gamma_5 D\phi_n = \lambda_n \phi_n, \quad \gamma_5 D(\Gamma_5\phi_n) = -\lambda_n(\Gamma_5\phi_n), \tag{2.16}$$

and the sum rule $n_+ + N_+ = n_- + N_-$ holds.

All the $n_\pm$ and $N_\pm$ states are the eigenstates of D, $D\phi_n = 0$ and $D\phi_n = (2/a)\phi_n$, respectively. If one denotes the number of states in (iii) by $2N_0$, the total number of states (dimension of the representation) N is given by $N = 2(n_+ + N_+ + N_0)$, which is expected to be a constant independent of background gauge field configurations.

3 The nature of the $N_\pm$ states with $\Gamma_5\phi_n = 0$

In the previous section we have seen that the consistency of the relation $Tr\gamma_5 = 0$ requires the presence of the $N_\pm$ states for an operator $\gamma_5 D$ satisfying (1.1) on a finite lattice. We want to show that the $N_\pm$ states are topological excitations associated with species doublers. For this purpose, we start with the conventional Wilson operator D_W

$$\begin{aligned}
D_W(n,m) &\equiv i\gamma^\mu C_\mu(n,m) + B(n,m) - \frac{1}{a}m_0\delta_{n,m}, \\
C_\mu(n,m) &= \frac{1}{2a}[\delta_{m+\mu,n}U_\mu(m) - \delta_{m,n+\mu}U^\dagger_\mu(n)], \\
B(n,m) &= \frac{r}{2a}\sum_\mu[2\delta_{n,m} - \delta_{m+\mu,n}U_\mu(m) - \delta_{m,n+\mu}U^\dagger_\mu(n)], \\
U_\mu(m) &= \exp[iagA_\mu(m)],
\end{aligned} \tag{3.1}$$

where we added a constant mass term to D_W. Our matrix convention is that γ^μ are anti-hermitian, $(\gamma^\mu)^\dagger = -\gamma^\mu$, and thus $\not{C} \equiv \gamma^\mu C_\mu(n,m)$ is hermitian

$$\not{C}^\dagger = \not{C}. \tag{3.2}$$

In the case of $\not{C}$, a very explicit construction of species doublers is known. For a square lattice one can explicitly show that the simplest lattice fermion action

$$S = \bar{\psi} i \not{C} \psi \tag{3.3}$$

is invariant under the transformation[16]

$$\psi' = \mathcal{T}\psi, \ \bar{\psi}' = \bar{\psi}\mathcal{T}^{-1} \tag{3.4}$$

where $\mathcal{T}$ stands for any one of the following 16 operators

$$1,\ T_1T_2,\ T_1T_3,\ T_1T_4,\ T_2T_3,\ T_2T_4,\ T_3T_4,\ T_1T_2T_3T_4, \tag{3.5}$$

and

$$T_1,\ T_2,\ T_3,\ T_4,\ T_1T_2T_3,\ T_2T_3T_4,\ T_3T_4T_1,\ T_4T_1T_2. \tag{3.6}$$

The operators T_μ are defined by

$$T_\mu \equiv \gamma_\mu \gamma_5 \exp\left(i\pi x^\mu/a\right) \tag{3.7}$$

and satisfy the relation

$$T_\mu T_\nu + T_\nu T_\mu = 2\delta_{\mu\nu} \tag{3.8}$$

with $T_\mu^\dagger = T_\mu = T_\mu^{-1}$ for anti-hermitian γ_μ. We denote the 16 operators by $\mathcal{T}_n,\ \ n = 0 \sim 15$, in the following with $\mathcal{T}_0 = 1$. By recalling that the operator T_μ adds the momentum π/a to the fermion momentum k_μ, we cover the entire Brillouin zone

$$-\frac{\pi}{2a} \le k_\mu < \frac{3\pi}{2a} \tag{3.9}$$

by the operation (3.4) starting with the free fermion defined in

$$-\frac{\pi}{2a} \le k_\mu < \frac{\pi}{2a}. \tag{3.10}$$

The operators in (3.5) commute with γ_5, whereas those in (3.6) anti-commute with γ_5 and thus change the sign of chiral charge, reproducing the 15 species doublers with correct chiral charge assignment; $\sum_{n=0}^{15}(-1)^n\gamma_5 = 0$.

In a smooth continuum limit, the operaor $\not{C}$ produces $\not{D}$ for each species doubler with alternating chiral charge. The relation $Tr\gamma_5 = 0$ for the operator $\not{C}$ is consistent for any background gauge field because of the presence of these species doublers, which are degenerate with the physical species in the present case.

The consistency of $Tr\gamma_5 = 0$ is generally analyzed by means of topological properties and thus it is best described in the nearly continuum limit. To be more precise, one may define the near continuum configurations by the momentum k_μ carried by the fermion

$$-\frac{\pi}{2a}\epsilon \le k_\mu \le \frac{\pi}{2a}\epsilon \tag{3.11}$$

for sufficiently small a and ϵ combined with the operation $\mathcal{T}_n$ in (3.5) and (3.6). To identify each species doubler clearly in the near continuum configurations, we also keep r/a and m_0/a finite for $a \rightarrow$ small [16], and the gauge fields are assumed to be sufficiently smooth. For these configurations, we can approximate the operator D_W by

$$D_W = i\not{D} + M_n + O(\epsilon^2) + O(agA_\mu) \tag{3.12}$$

for each species doubler, where the mass parameters M_n stand for $M_0 = -\frac{m_0}{a}$ and one of

$$\begin{array}{llll} \frac{2r}{a} - \frac{m_0}{a}, & (4,-1); & \frac{4r}{a} - \frac{m_0}{a}, & (6,1) \\ \frac{6r}{a} - \frac{m_0}{a}, & (4,-1); & \frac{8r}{a} - \frac{m_0}{a}, & (1,1) \end{array} \tag{3.13}$$

for $n = 1 \sim 15$[2]. Here we denoted (multiplicity, chiral charge) in the bracket for species doublers. In (3.12) we used the relation valid for the configurations (3.11), for example,

$$\begin{aligned} D_W(k) &= \sum_\mu \gamma^\mu \frac{\sin ak_\mu}{a} + \frac{r}{a}\sum_\mu(1-\cos ak_\mu) - \frac{m_0}{a} \\ &= \gamma^\mu k_\mu(1+O(\epsilon^2)) + \frac{r}{a}O(\epsilon^2) - \frac{m_0}{a} \end{aligned} \tag{3.14}$$

in the momentum representation with vanishing gauge field.

In these near continuum configurations, the topological properties are specified by the operator $\not{D}$ in D_W. The overlap operator D introduced by Neuberger[2] , which satisfies the relation (1.1), has an explicit expression

$$aD = 1 - \gamma_5 \frac{H}{\sqrt{H^2}} = 1 + D_W \frac{1}{\sqrt{D_W^\dagger D_W}} \tag{3.15}$$

where $D_W = -\gamma_5 H$ is the Wilson operator. For the near continuum configurations specified above in (3.11), one can approximate

$$\begin{aligned} D &= \sum_{n=0}^{15} (1/a)[1 + (i\not{D} + M_n)\frac{1}{\sqrt{\not{D}^2 + M_n^2}}]|n\rangle\langle n|, \\ \gamma_5 D &= \sum_{n=0}^{15} (-1)^n \gamma_5 (1/a)[1 + (i\not{D} + M_n)\frac{1}{\sqrt{\not{D}^2 + M_n^2}}]|n\rangle\langle n|, \\ \gamma_5 &= \sum_{n=0}^{15} (-1)^n \gamma_5 |n\rangle\langle n|. \end{aligned} \tag{3.16}$$

Here we explicitly write the projection $|n\rangle\langle n|$ for each species doubler. The operators in (3.16) preserve the Ginsparg-Wilson relation (1.1).

The above expression of D in (3.16) shows that

$$\begin{aligned} D\phi_l &= 0, \\ D\phi_l &= \frac{2}{a}\phi_l \end{aligned} \tag{3.17}$$

for the physical species and the unphysical species doublers, respectively, if one uses the *zero-modes* in

$$\not{D}\phi_l = \lambda_n \phi_n. \tag{3.18}$$

Note that $M_0 < 0$ and the rest of $M_n > 0$ in (3.13) and (3.16) [2]. We also note that ϕ_l can be a simultaneous eigenstate of γ_5 only for $\not{D}\phi_l = 0$. Namely, the $N_\pm$ states with eigenvalue $2/a$ of D in fact correspond to *topological excitations* associated with species doublers. This means that the multiplicities of these $N_\pm$ are quite high due to the 15 species doublers, although they satisfy the sum rule $n_+ + N_+ = n_- + N_-$: This sum rule itself is a direct consequence of (3.17), (3.18) and $\sum_n (-1)^n \gamma_5 = 0$ if one recalls $\phi_l^\dagger \gamma_5 \phi_l = 0$ for $\lambda_l \neq 0$. The relation (3.17) with zero-modes in (3.18) shows that the index relation (2.10) stands for a lattice version of the Atiyah-Singer index theorem.

We here briefly sketch an evaluation of the anomaly, since it justifies our analysis based on the effective expressions in (3.16). For an operator $O(x,y)$ defined on the lattice, one may define

$$O_{mn} \equiv \sum_{x,y} \phi_m^*(x) O(x,y) \phi_n(y), \tag{3.19}$$

and the trace

$$
\begin{aligned}
TrO &= \sum_n O_{nn} \\
&= \sum_n \sum_{x,y} \phi_n^*(x) O(x,y) \phi_n(y) \\
&= \sum_x (\sum_{n,y} \phi_n^*(x) O(x,y) \phi_n(y)). \qquad (3.20)
\end{aligned}
$$

The local version of the trace (or anomaly) is then defined by $trO(x,x) \equiv \sum_{n,y} \phi_n^*(x) O(x,y) \phi_n(y)$. For the operator of our interest, we have

$$
tr\gamma_5(1 - \frac{a}{2}D)(x) = -\frac{1}{2}\sum_{n=0}^{15} tr \int_{-\frac{\pi}{2a}}^{\frac{\pi}{2a}} \frac{d^4k}{(2\pi)^4} e^{-ikx} \mathcal{T}_n^{-1} \gamma_5 \frac{D_W}{\sqrt{D_W^\dagger D_W}} \mathcal{T}_n e^{ikx} \qquad (3.21)
$$

where we used the plane wave basis defined in (3.10) combined with the operation $\mathcal{T}_n$. We also used the chiral charge assignment (3.16) and a short hand notation $Oe^{ikx} = \sum_y O(x,y)e^{iky}$.

We first take the $a \to 0$ limit of (3.21) with all $M_n, n = 0 \sim 15$, kept fixed and then take the limit $|M_n| \to \infty$ later. For fixed M_n (to be precise, for fixed m_0/a and r/a), one can confirm that the above integral (3.21) for the domain $\frac{\pi}{2a}\epsilon \le |k_\mu| \le \frac{\pi}{2a}$ vanishes (at least) linearly in a for $a \to 0$, if one takes into account the trace with γ_5. See also Refs.[7][9]. In the remaining integral

$$
-\frac{1}{2}\sum_{n=0}^{15}(-1)^n tr \int_{-\frac{\pi}{2a}\epsilon}^{\frac{\pi}{2a}\epsilon} \frac{d^4k}{(2\pi)^4} e^{-ikx} \gamma_5 \mathcal{T}_n^{-1} D_W \frac{1}{\sqrt{D_W^\dagger D_W}} \mathcal{T}_n e^{ikx} \qquad (3.22)
$$

one may take the limit $a \to 0$ (and $\frac{\pi}{2a}\epsilon \to \infty$) with letting ϵ arbitrarily small. By taking (3.12) into account, one thus recovers the local anomaly [8]

$$
\begin{aligned}
&-\frac{1}{2}\sum_{n=0}^{15}(-1)^n tr \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} e^{-ikx} \gamma_5 (i \not{D} + M_n) \frac{1}{\sqrt{\not{D}^2 + M_n^2}} e^{ikx} \\
&= \frac{1}{2} tr \int_{-\infty}^{\infty} \frac{d^4k}{(2\pi)^4} e^{-ikx} \gamma_5 [\frac{1}{\sqrt{\not{D}^2/M_0^2 + 1}} - \sum_{n=1}^{15}(-1)^n \frac{1}{\sqrt{\not{D}^2/M_n^2 + 1}}] e^{ikx} \\
&= \frac{g^2}{32\pi^2} tr \epsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \qquad (3.23)
\end{aligned}
$$

for $|M_n| \to \infty$. The domain in (3.11) with arbitrarily small but finite ϵ thus correctly describes the topological aspects of the continuum limit in the present prescription.

Here we went through some details of the anomaly calculation to show that the interpretation of the $N_\pm$ states in (2.15) as topological excitations associated with species doublers, as is shown in (3.17), is also consistent with the local anomaly calculation. As for a general analysis of chiral anomaly in the overlap operator, see Ref.[17].

3.1 General lattice Dirac operator and $Tr\gamma_5 = 0$

We expect that our analysis of $Tr\gamma_5 = 0$, namely its consistency is ensured only by the presence of the would-be species doublers in the Hilbert space, works for a general lattice Dirac operator, since any lattice operator contains $\not{C}$ as an essential part. For the smooth near continuum configurations, the lowest dimensional operator $\not{C}$ is expected to specify the topological properties.

We also note that the Pauli-Villars regularization in continuum theory can be analyzed in a similar way. The Pauli-Villars regulator is defined in the path integral by introducing a bosonic spinor ϕ into the action

$$S = \int d^4x[\bar{\psi}(i \not{D} - m)\psi + \bar{\phi}(i \not{D} - M)\phi]. \tag{3.24}$$

The Jacobian for the global chiral transformation then gives rise to the graded trace[11]

$$Tr\gamma_5 = Tr_\psi\gamma_5 - Tr_\phi\gamma_5 = 0. \tag{3.25}$$

The relation $Tr\gamma_5 = 0$ is thus consistent with any topologically non-trivial background gauge field because of the presence of the unphysical regulator ϕ. This ϕ is analogous to the species doublers in lattice regularization.

4 Implications of the present analysis

We have shown that the consistency of $Tr\gamma_5 = 0$ for topologically non-trivial background gauge fields requires the presence of some unphysical states in the Hilbert space. Coming back to the original lattice theory defined by

$$S = \bar{\psi}D\psi \tag{4.1}$$

with D satisfying the relation (1.1), one obtains twice of (2.10) as a Jacobian factor for the global chiral transformation [5] $\delta\psi = i\epsilon\gamma_5(1 - \frac{a}{2}D)\psi$ and $\delta\bar{\psi} = \bar{\psi}i\epsilon(1 - \frac{a}{2}D)\gamma_5$, which leaves the action (4.1) invariant. One can rewrite (2.10) as

$$Tr\gamma_5(1 - \frac{a}{2}D) = \tilde{T}r\gamma_5(1 - \frac{a}{2}D) = \tilde{T}r\gamma_5 = n_+ - n_- \tag{4.2}$$

where the modified trace $\tilde{T}r$ is defined by truncating the unphysical $N_\pm$ states with $\lambda_n = \pm 2/a$. Without the $N_\pm$ states, $\tilde{T}r\gamma_5\frac{a}{2}D = 0$ since the eigenvalues λ_n of $\gamma_5 D$ with $\lambda_n \neq 0, \pm 2/a$ appear always pairwise at $\pm|\lambda_n|$.

If one takes a smooth continuum limit of $\tilde{T}r\gamma_5 = n_+ - n_-$ in (4.2) , one recovers the result of the continuum path integral (1.4). If one considers that $\tilde{T}r\gamma_5$ is too abstract, one may define it more concretely by

$$\begin{aligned} Tr\gamma_5(1 - \frac{a}{2}D)f(\frac{(\gamma_5 D)^2}{M^2}) &= \tilde{T}r\gamma_5(1 - \frac{a}{2}D)f(\frac{(\gamma_5 D)^2}{M^2}) \\ &= \tilde{T}r\gamma_5 f(\frac{(\gamma_5 D)^2}{M^2}) = n_+ - n_- \end{aligned} \tag{4.3}$$

for *any* $f(x)$ which rapidly goes to 0 for $x \to \infty$ with $f(0) = 1$. This relation suggests that we can extract the local index [10] (or anomaly) by

$$tr\gamma_5(1 - \frac{a}{2}D)f(\frac{(\gamma_5 D)^2}{M^2})(x) \tag{4.4}$$

which is shown to be independent of the choice of $f(x)$ in the limit $a \to 0$ and leads to (3.23) by using only the general properties of D [8]. We thus naturally recover the result of the continuum path integral[11]. As for an interesting algebraic analysis of the local anomaly $tr\gamma_5(1 - \frac{a}{2}D)$, see [18].

As for a more practical implication of $Tr\gamma_5 = 0$ in lattice theory, one may say that any result which depends explicitly on the presence of the states $N_\pm$ is *unphysical.* It is thus necessary to define the scalar density (or mass term) and pseudo-scalar density in the theory (4.1) by [19][6]

$$\begin{aligned} S(x) &= \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L = \bar{\psi}(1 - \frac{a}{2}D)\psi, \\ P(x) &= \bar{\psi}_L\psi_R - \bar{\psi}_R\psi_L = \bar{\psi}\gamma_5(1 - \frac{a}{2}D)\psi. \end{aligned} \tag{4.5}$$

Here we defined two independent projection operators

$$\begin{aligned} P_\pm &= \frac{1}{2}(1 \pm \gamma_5) \\ \hat{P}_\pm &= \frac{1}{2}(1 \pm \hat{\gamma}_5) \end{aligned} \tag{4.6}$$

with $\hat{\gamma}_5 = \gamma_5(1 - aD)$ which satisfies $\hat{\gamma}_5^2 = 1$ [6]. The left- and right- components are then defined by

$$\bar{\psi}_{L,R} = \bar{\psi}P_\pm, \quad \psi_{R,L} = \hat{P}_\pm\psi \tag{4.7}$$

which is based on the decomposition

$$D = P_+ D\hat{P}_- + P_- D\hat{P}_+. \tag{4.8}$$

The physical operators $S(x)$ and $P(x)$ in (4.5) do not contain the contribution from the unphysical states $N_\pm$ in (2.15). In the spirit of this construction, the definition of the index by (4.3) which is independent of unphysical states $N_\pm$ is natural. In particular, all the unphysical species doublers (not only the topological ones at $2/a$) decouple from the anomaly defined by (4.4) in the limit $a \to 0$ with fixed M.

The customary calculation of the index (and also anomaly) by using the relation[4]-[7][9]

$$Tr\gamma_5(1 - \frac{a}{2}D) = Tr(-\frac{a}{2}\gamma_5 D) = n_+ - n_- \tag{4.9}$$

by itself is of course consistent, since one simply includes the unphysical states $N_\pm$ in evaluating $Tr\gamma_5 = 0$, and consequently one obtains the index $Tr(-\frac{a}{2}\gamma_5 D)$ from the unphysical states $N_\pm$ only. We after all know that the left-hand side of (4.9) is independent of $N_\pm$.

Rather, the major message of our analysis is that the continuum limit of $Tr\gamma_5 = 0$ in (1.6) (unlike the relation $\tilde{T}r\gamma_5 = n_+ - n_-$) *cannot* be defined in a consistent way when the (would-be) species doublers disappear from the Hilbert space. It is clear from the expression of $Tr\gamma_5 = 0$ in (1.6) that the $a \to 0$ limit of $Tr\gamma_5 = 0$ is not defined consistently. One may then ask how the calculation of local anomaly on the basis of (4.9) could be consistent in the limit $a \to 0$ if $Tr\gamma_5 = 0$ is inconsistent? A key to resolve this apparent paradox is the failure of the decoupling of heavy fermions in the evaluation of anomaly. The massive unphysical species doublers do not decouple from the anomaly , as is seen in (3.23), for example. If one insists on $Tr\gamma_5 = 0$ in the continuum limit, one is also insisting on the failure of the decoupling of these infinitely massive particles from $Tr\gamma_5 = 0$. The contributions of these heavy fermions to the anomaly and to $Tr\gamma_5 = 0$ precisely cancel, just as in the case of the evaluation of global index in (4.9). Namely, the local anomaly itself is *independent* of these massive species doublers in the continuum limit, as is clear in (4.4). It is an advantage of the finite lattice formulation that we can now clearly illustrate this subtle cancellation of the contributions of those ultra-heavy $N_\pm$-states to $Tr\gamma_5 = 0$ and anomaly on the basis of (1.6).

The unphysical nature of the $N_\pm$-states has been recently clarified from a different view point in Re.[20]

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